ROTOR-ROUTER AGGREGATION ON THE LAYERED SQUARE LATTICE

WOUTER KAGER AND LIONEL LEVINE

Abstract. In rotor-router aggregation on the square lattice $\mathbb{Z}^2$, particles starting at the origin perform deterministic analogues of random walks until reaching an unoccupied site. The limiting shape of the cluster of occupied sites is a disk. We consider a small change to the routing mechanism for sites on the x- and y-axes, resulting in a limiting shape which is a diamond instead of a disk. We show that for a certain choice of initial rotors, the occupied cluster grows as a perfect diamond.

1. Introduction

Recently there has been considerable interest in low-discrepancy deterministic analogues of random processes. An example is rotor-router walk [PDDK96], a deterministic analogue of random walk. Based at every vertex of the square grid $\mathbb{Z}^2$ is a *rotor* pointing to one of the four neighboring vertices. A chip starts at the origin and moves in discrete time steps according to the following rule. At each time step, the rotor based at the location of the chip turns clockwise 90 degrees, and the chip then moves to the neighbor to which that rotor points.

Holroyd and Propp [HP09] show that rotor-router walk captures the mean behavior of random walk in a variety of respects: stationary measure, hitting probabilities and hitting times. Cooper and Spencer [CS06] study rotor-router walks in which n chips starting at arbitrary even vertices each take a fixed number t of steps, showing that the final locations of the chips approximate the distribution of a random walk run for t steps to within constant error independent of n and t. Rotor-router walk and other low-discrepancy deterministic processes have algorithmic applications in areas such as broadcasting information in networks [DFS08] and iterative load-balancing [FGS10]. The common theme running through these results is that the deterministic process captures some aspect of the mean behavior of the random process, but with significantly smaller fluctuations than the random process.

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Rotor-router aggregation is a growth model defined by repeatedly releasing chips from the origin $o \in \mathbb{Z}^2$, each of which performs a rotor-router walk until reaching an unoccupied site. Formally, we set $A_0 = \{o\}$ and recursively define

$$A_{m+1} = A_m \cup \{z_m\} \tag{1}$$

for $m \geq 0$, where z_m is the endpoint of a rotor-router walk started at the origin in $\mathbb{Z}^2$ and stopped on exiting A_m. We do not reset the rotors when a new chip is released.

It was shown in [LP08, LP09] that for any initial rotor configuration, the asymptotic shape of the set A_m is a Euclidean disk. It is in some sense remarkable that a growth model defined on the square grid, and without any reference to the Euclidean norm $|x| = (x_1^2 + x_2^2)^{1/2}$, nevertheless has a circular limiting shape. Here we investigate the dependence of this shape on changes to the rotor-router mechanism.

The *layered square lattice* $\hat{\mathbb{Z}}^2$ is the directed multigraph obtained from the usual square grid $\mathbb{Z}^2$ by reflecting all directed edges on the x- and y-axes that point to a vertex closer to the origin. For example, for each positive integer n, the edge from $(n, 0)$ to $(n-1, 0)$ is reflected so that it points from $(n, 0)$ to $(n+1, 0)$. Only edges on the x- and y-axes are affected. Rotor-router walk on $\hat{\mathbb{Z}}^2$ is equivalent to rotor-router walk on $\mathbb{Z}^2$ with one modification: if the chip is on one of the axes, and the rotor points along the axis towards the origin after it is turned, then the chip ignores the rotor and moves in the opposite direction instead.

For $n \geq 0$, let

$$D_n = \left\{(x, y) \in \mathbb{Z}^2 \,:\, |x| + |y| \leq n\right\}.$$

We call D_n the *diamond of radius n*. Our main result is the following.

Theorem 1. *There is a rotor configuration ρ_0, such that rotor-router aggregation $(A_m)_{m\geq 0}$ on $\hat{\mathbb{Z}}^2$ with rotors initially configured as ρ_0 satisfies*

$$A_{2n(n+1)} = D_n \qquad \text{for all } n \geq 0.$$

A formal definition of rotor-router walk on $\hat{\mathbb{Z}}^2$ and an explicit description of the rotor configuration ρ_0 are given below.

Let us remark on two features of Theorem 1. First, note that the rotor mechanism on $\hat{\mathbb{Z}}^2$ is identical to that on $\mathbb{Z}^2$ except for sites on the x- and y-axes. Nevertheless, changing the mechanism on the axes completely changes the limiting shape, transforming it from a disk into a diamond. Second, not only is the aggregate close to a diamond, it is exactly equal to a diamond whenever it has the appropriate size (Figure 1).

In [KL10], we studied the analogous stochastic growth model, known as *internal DLA*, defined by the growth rule (1) using random walk on $\hat{\mathbb{Z}}^2$. This random walk has a *uniform layering property*: at any fixed time, its distribution is a mixture of uniform distributions on the diamond layers

$$L_m = \left\{(x, y) \in \mathbb{Z}^2 \,:\, |x| + |y| = m\right\}, \qquad m \geq 1.$$

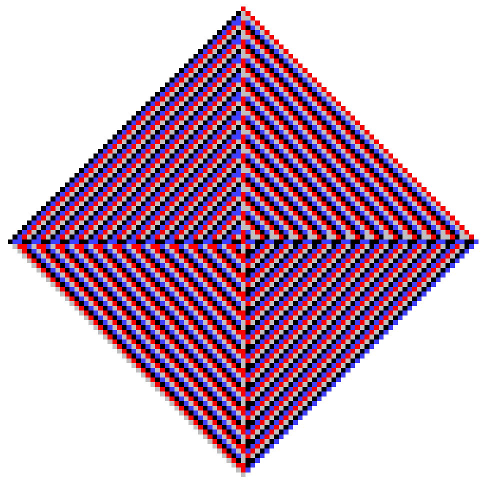

FIGURE 1. The rotor-router aggregate of 5101 chips in the layered square lattice $\hat{\mathbb{Z}}^2$ is a perfect diamond of radius 50. The colors encode the directions of the final rotors at the occupied vertices: red = north, blue = east, gray = south and black = west.

It is for this reason that we call $\hat{\mathbb{Z}}^2$ the layered square lattice.

As a consequence of the uniform layering property, internal DLA on $\hat{\mathbb{Z}}^2$ also grows as a diamond, but with random fluctuations at the boundary. Theorem 1 thus represents an extreme of discrepancy reduction: passing to the deterministic analogue removes *all* of the fluctuations from the random process, leaving only the mean behavior. For a similar "no discrepancy" result when the underlying graph is a regular tree instead of $\hat{\mathbb{Z}}^2$, see [LL09].

To formally define rotor-router walk on $\hat{\mathbb{Z}}^2$, write $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and let $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ be clockwise rotation by 90 degrees. For each site $z \in \mathbb{Z}^2 \setminus \{o\}$ there is a unique choice of a number $j \in \{0, 1, 2, 3\}$ and a point w in the quadrant

$$Q = \big\{(x, y) \in \mathbb{Z}^2 \,:\, x \geq 0,\, y > 0\big\}$$

such that $z = R^j w$. Given j and $w = (x, y)$, we associate to $z = R^j w$ a 4-tuple $(e_z^0, e_z^1, e_z^2, e_z^3)$ of directed outgoing edges, where

$$e_z^i = \begin{cases} (z, z + R^j \mathbf{e}_2) & \text{if } i = 2 \text{ and } x = 0; \\ (z, z + R^{i+j} \mathbf{e}_2) & \text{otherwise.} \end{cases} \tag{2}$$

Thus, for $z \in Q$ (hence $j = 0$ and $w = z$) the edges $e_z^0, e_z^1, e_z^2, e_z^3$ point respectively north, east, north, west when z is on the y-axis; and north, east, south, west when z is off the y-axis. For z in another quadrant, the directions of $e_z^0, e_z^1, e_z^2, e_z^3$ are obtained using rotational symmetry. To the origin we associate the 4-tuple $(e_o^0, e_o^1, e_o^2, e_o^3)$ where $e_o^i = (o, R^i \mathbf{e}_2)$ for $i = 0, 1, 2, 3$.

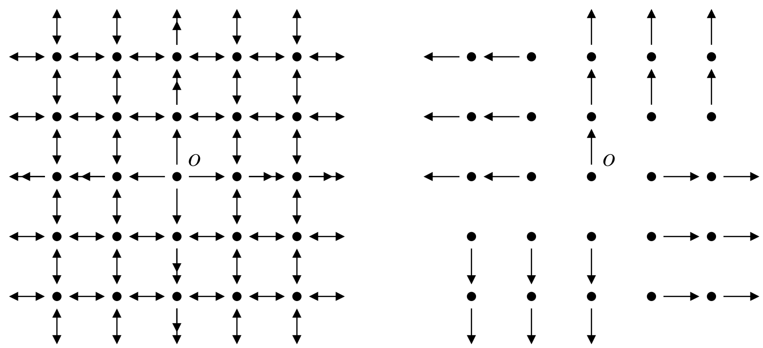

FIGURE 2. Left: The layered square lattice $\hat{\mathbb{Z}}^2$. Each directed edge is represented by an arrow; multiple edges on the x- and y-axes are represented by double arrows. The origin o is in the center. Right: The initial rotor configuration ρ_0.

For every $z \in \mathbb{Z}^2$, let E_z be the multiset $\{e_z^0, e_z^1, e_z^2, e_z^3\}$. If $e = e_z^i \in E_z$, we denote by e^+ the next element $e_z^{i+1 \bmod 4}$ of E_z under the cyclic shift. The *layered square lattice* $\hat{\mathbb{Z}}^2$ is the directed multigraph with vertex set $V = \mathbb{Z}^2$ and edge multiset $E = \bigcup_{z \in \mathbb{Z}^2} E_z$, where edges that appear twice in E_z have multiplicity two (Figure 2, left). Thus every vertex has out-degree four, and every vertex except for the origin and its neighbors has in-degree four.

The initial rotor configuration ρ_0 appearing in Theorem 1 is given by

$$\rho_0(z) = e_z^0, \qquad z \in \mathbb{Z}^2. \tag{3}$$

It has every rotor in the quadrant Q pointing north, and rotor directions in the other quadrants given by rotational symmetry (Figure 2, right).

We may now describe rotor-router walk on $\hat{\mathbb{Z}}^2$ as follows. Given a rotor configuration ρ with a chip at vertex z, a single step of the walk consists of changing the rotor $\rho(z)$ to $\rho(z)^+$, and moving the chip to the vertex pointed to by the new rotor $\rho(z)^+$. This yields a new rotor configuration and a new chip location. Note that if the walk visits z infinitely many times, then it visits all out-neighbors of z infinitely many times, and hence visits every vertex of $\hat{\mathbb{Z}}^2$ (except for o) infinitely many times. It follows that rotor-router walk exits any finite subset of $\hat{\mathbb{Z}}^2$ in a finite number of steps; in particular, rotor-router aggregation terminates in finitely many steps.

2. Strong Abelian Property

In this section we prove a "strong abelian property" of the rotor-router model, Theorem 2, which holds on any finite directed multigraph. To prove Theorem 1, we will apply the results of this section to the induced subgraph D_n of $\hat{\mathbb{Z}}^2$.

Let $G = (V, E)$ be a finite directed multigraph (it may have loops and multiple edges). Each edge $e \in E$ is directed from its source vertex $\mathtt{s}(e)$ to

its target vertex $\mathtt{t}(e)$. For a vertex $v \in V$, write

$$E_v = \{e \in E \,:\, \mathtt{s}(e) = v\}$$

for the multiset of edges emanating from v. The *outdegree* d_v of v is the cardinality of E_v.

Fix a nonempty subset $S \subset V$ of vertices called *sinks*. Let $V' = V \setminus S$, and for each vertex $v \in V'$, fix a numbering $e_v^0, \ldots, e_v^{d_v-1}$ of the edges in E_v. If $e = e_v^i \in E_v$, we denote by e^+ the next element $e_v^{i+1 \bmod d_v}$ of E_v under the cyclic shift.

A *rotor configuration* on G is a function

$$\rho : V' \to E$$

such that $\rho(v) \in E_v$ for all $v \in V'$. A *chip configuration* on G is a function

$$\sigma : V \to \mathbb{Z}.$$

Note that we do not require $\sigma \geq 0$. If $\sigma(v) = m > 0$, we say there are m chips at vertex v; if $\sigma(v) = -m < 0$, we say there is a hole of depth m at vertex v.

Fix a vertex $v \in V'$. Given a rotor configuration ρ and a chip configuration σ, the operation F_v of *firing* v yields a new pair

$$F_v(\rho, \sigma) = (\rho', \sigma')$$

where

$$\rho'(w) = \begin{cases} \rho(w)^+ & \text{if } w = v; \\ \rho(w) & \text{if } w \neq v; \end{cases}$$

and

$$\sigma'(w) = \begin{cases} \sigma(w) - 1 & \text{if } w = v; \\ \sigma(w) + 1 & \text{if } w = \mathtt{t}(\rho(v)^+); \\ \sigma(w) & \text{otherwise.} \end{cases}$$

In words, F_v first rotates the rotor at v, then sends a single chip from v along the new rotor $\rho(v)^+$. We do not require $\sigma(v) > 0$ in order to fire v. Thus if $\sigma(v) = 0$, i.e., no chips are present at v, then firing v will create a hole of depth 1 at v; if $\sigma(v) < 0$, so that there is already a hole at v, then firing v will increase the depth of the hole by 1.

Observe that the firing operators commute: $F_v F_w = F_w F_v$ for all $v, w \in V'$. Denote by $\mathbb{N}$ the nonnegative integers. Given a function

$$u : V' \to \mathbb{N}$$

we write

$$F^u = \prod_{v \in V'} F_v^{u(v)}$$

where the product denotes composition. By commutativity, the order of the composition is immaterial.

A rotor configuration ρ is *acyclic* if the spanning subgraph $(V, \rho(V'))$ has no directed cycles or, equivalently, if for every nonempty subset $A \subset V'$ there is a vertex $v \in A$ such that $\mathtt{t}(\rho(v)) \notin A$.

In the following theorem and lemmas, for functions f, g defined on a set of vertices $A \subset V$, we write "$f = g$ on A" to mean that $f(v) = g(v)$ for all $v \in A$, and "$f \leq g$ on A" to mean that $f(v) \leq g(v)$ for all $v \in A$.

Theorem 2 (Strong Abelian Property)**.** *Let ρ be a rotor configuration and σ a chip configuration on G. Given two functions $u_1, u_2 : V' \to \mathbb{N}$, write*

$$F^{u_i}(\rho, \sigma) = (\rho_i, \sigma_i), \qquad i = 1, 2.$$

If $\sigma_1 = \sigma_2$ on V', and both ρ_1 and ρ_2 are acyclic, then $u_1 = u_2$.

Note that the equality $u_1 = u_2$ implies that $\rho_1 = \rho_2$, and that $\sigma_1 = \sigma_2$ on all of V. For a similar idea with an algorithmic application, see [FL10, Theorem 1].

In a typical application of Theorem 2, we take $\sigma_1 = \sigma_2 = 0$ on V', and u_1 to be the usual rotor-router odometer function

$$u_1(v) = \#\{1 \leq j \leq k \,:\, v_j = v\}$$

where $v_1, v_2, \ldots, v_k$ is a *complete legal firing sequence* for the initial configuration (ρ, σ); that is, a sequence of vertices which, when fired in order, causes all chips to be routed to the sinks without ever creating any holes. Provided $u_1(v) > 0$ for all $v \in V'$, the resulting rotor configuration ρ_1 is acyclic: indeed, for any nonempty subset $A \subset V'$, the rotor at the last vertex of A to fire points to a vertex not in A.

The usual abelian property of rotor-router walk [DF91, Theorem 4.1] says that any two complete legal firing sequences have the same odometer function. The Strong Abelian Property allows us to drop the hypothesis of legality: any two complete firing sequences whose final rotor configurations are acyclic have the same odometer function, *even if one or both of these firing sequences temporarily creates holes.*

In our application to rotor-router aggregation on the layered square lattice, we take $V = D_n$ and $S = L_n$. We will take σ to be the chip configuration consisting of $2n(n+1)+1$ chips at the origin, and ρ to be the initial rotor configuration ρ_0. Letting the chips at the origin in turn perform rotor-router walk until finding an unoccupied site defines a legal firing sequence (although not a complete one, since not all chips reach the sinks). In the next section, we give an explicit formula for the corresponding odometer function, and use Theorem 2 to prove its correctness. The proof of Theorem 1 is completed by showing that each nonzero vertex in D_n receives exactly one more chip from its neighbors than the number of times it fires.

To prove Theorem 2 we start with the following lemma.

Lemma 3. *Let $u : V' \to \mathbb{N}$, and write*

$$F^u(\rho, \sigma) = (\rho_1, \sigma_1).$$

If $\sigma = \sigma_1$, and ρ_1 is acyclic, then $u = 0$.

Proof. Let $A = \{v \in V' : u(v) > 0\}$, and suppose that A is nonempty. Since ρ_1 is acyclic, there is a vertex $v \in A$ whose rotor $\rho_1(v)$ points to a vertex not in A. The final time v is fired, it sends a chip along this rotor; thus, at least one chip exits A. Since the vertices not in A do not fire, no chips enter A, hence

$$\sum_{v \in A} \sigma_1(v) < \sum_{v \in A} \sigma(v)$$

contradicting $\sigma = \sigma_1$. Therefore, A is empty. □

Theorem 2 follows immediately from the next lemma.

Lemma 4. *Let $u_1, u_2 : V' \to \mathbb{N}$, and write*

$$F^{u_i}(\rho, \sigma) = (\rho_i, \sigma_i), \qquad i = 1, 2.$$

If ρ_1 is acyclic and $\sigma_2 \le \sigma_1$ on V', then $u_1 \le u_2$ on V'.

Proof. Let

$$(\hat{\rho}, \hat{\sigma}) = F^{\min(u_1, u_2)}(\rho, \sigma).$$

Then (ρ_1, σ_1) is obtained from $(\hat{\rho}, \hat{\sigma})$ by firing only vertices in the set $A = \{v \in V' : u_1(v) > u_2(v)\}$, so

$$\hat{\sigma} \le \sigma_1 \text{ on } A^c.$$

Likewise, (ρ_2, σ_2) is obtained from $(\hat{\rho}, \hat{\sigma})$ by firing only vertices in A^c, so

$$\hat{\sigma} \le \sigma_2 \le \sigma_1 \text{ on } A.$$

Thus $\hat{\sigma} \le \sigma_1$ on V. Since $\sum_{v \in V} \hat{\sigma}(v) = \sum_{v \in V} \sigma_1(v)$ it follows that $\hat{\sigma} = \sigma_1$. Taking

$$u = u_1 - \min(u_1, u_2)$$

in Lemma 3, since $F^u(\hat{\rho}, \hat{\sigma}) = (\rho_1, \sigma_1)$ we conclude that $u = 0$. □

3. Proof of Theorem 1

Consider again the rotor-router model on the layered square lattice $\hat{\mathbb{Z}}^2$. We will work with the induced subgraph D_n of $\hat{\mathbb{Z}}^2$, taking the sites in the outermost layer L_n as sinks.

Recall our notation

$$Q = \{(x, y) \in \mathbb{Z}^2 : x \ge 0, y > 0\}$$

for the first quadrant of $\mathbb{Z}^2$. We have $\mathbb{Z}^2 = \{o\} \cup \left(\bigcup_{i=0}^3 R^i Q\right)$, where $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is clockwise rotation by 90 degrees. Fix n, and for $z = (x, y) \in D_n$ write

$$\ell_z = n - |x| - |y|.$$

Consider the sets

$$C_2 = \{(x, y) \in Q \cap D_{n-1} : x > 0,\, y \ge 2,\, \ell_{(x,y)} \equiv 2 \bmod 4\}$$

$$C_3 = \{(x, y) \in Q \cap D_{n-1} : x > 0,\, y \ge 1,\, \ell_{(x,y)} \equiv 3 \bmod 4\}$$

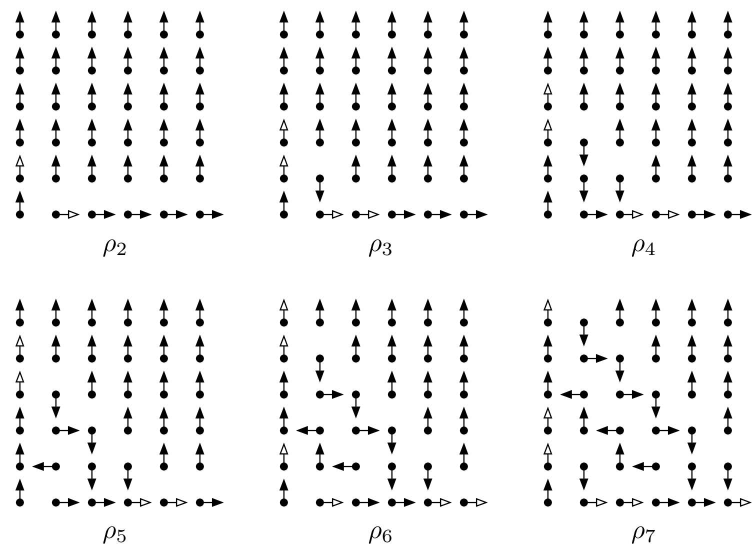

FIGURE 3. The rotor configurations $\rho_2, \rho_3, \ldots, \rho_7$ in the first quadrant. The lower left corner is the origin in each picture. On the axes, the black arrows correspond to the directed edge e_z^0 in (2), and open-headed arrows to e_z^2.

and

$$C = \bigcup_{i=0}^{3} R^i(C_2 \cup C_3).$$

Define $u_n : D_{n-1} \to \mathbb{N}$ by

$$u_n = u_n' - 1_C \tag{4}$$

where

$$u_n'(z) = \begin{cases} 2n(n+1) & \text{if } z = o; \\ \ell_z(\ell_z + 1) & \text{if } z \neq o; \end{cases} \tag{5}$$

and $1_C(z)$ is the indicator function which is 1 for $z \in C$ and 0 for $z \notin C$.

Let ρ_0 be the initial rotor configuration (3), and define the rotor configuration ρ_n on D_{n-1} and chip configuration σ_n on D_n by setting

$$F^{u_n}(\rho_0, (2n^2 + 2n + 1)\delta_o) = (\rho_n, \sigma_n).$$

From the formula (4) it is easy to obtain an explicit description of ρ_n, and to verify that these rotor configurations are acyclic for all $n \geq 1$. Figure 3 depicts the rotor configurations $\rho_2, \rho_3, \ldots, \rho_7$ in the first quadrant.

Lemma 5. *For all $n \geq 1$, we have $\sigma_n = 1_{D_n}$.*

Proof. The origin o has no incoming edges in $\hat{\mathbb{Z}}^2$, so it receives no chips from its neighbors. Since $u_n(o) = 2n^2 + 2n$, the origin is left with exactly one chip after firing. The sink vertices L_n do not fire and only receive chips. Since

$u_n(z) = 2$ for all $z \in L_{n-1}$, it follows from (2) and (3) that exactly one chip is sent to each sink vertex.

It remains to show that $\sigma_n(z) = 1$ for each vertex $z \in D_{n-1} \setminus \{o\}$, i.e., that the number of chips sent to z by its neighbors is one more than the number of times z is fired (that is, $1 + u_n(z)$). To show this, write

$$F^{u'_n}(\rho_0, (2n^2 + 2n + 1)\delta_o) = (\rho'_n, \sigma'_n)$$

where u'_n is given by (5). We will argue that $\sigma'_n(z) = 1$ and that $\sigma_n(z) = \sigma'_n(z)$. By symmetry, it suffices to consider points $z = (x, y)$ in $D_{n-1} \cap Q$. We argue separately in the two cases $x = 0$ and $x > 0$ (on the axis and off the axis).

Case 1: $x = 0$. Under $F^{u'_n}$, the site z fires $\ell_z(\ell_z + 1)$ times. If $y > 1$, its neighbor $z - \mathbf{e}_2$ fires $(\ell_z + 1)(\ell_z + 2)$ times, and from (2) and (3) we see that it sends a chip to z every even time it is fired. Since $(\ell_z + 1)(\ell_z + 2)$ is even, it follows that $z - \mathbf{e}_2$ sends $\frac{1}{2}(\ell_z + 1)(\ell_z + 2)$ chips to z. The same is true if $y = 1$, since then $\ell_z = n - 1$, and the origin $o = z - \mathbf{e}_2$ sends $\frac{1}{2}n(n + 1)$ chips to z.

The only other vertices that send chips to z under $F^{u'_n}$ are its left and right neighbors $z \pm \mathbf{e}_1$. Since $\ell_{z \pm \mathbf{e}_1} = \ell_z - 1$, these neighbors fire $\ell_z(\ell_z - 1)$ times. We claim that together they send $\frac{1}{2}\ell_z(\ell_z - 1)$ chips to z. To see this, note that if we fire these two vertices in parallel, they send one chip to z every two times we fire. We therefore conclude that

$$\sigma'_n(z) = \tfrac{1}{2}(\ell_z + 1)(\ell_z + 2) + \tfrac{1}{2}\ell_z(\ell_z - 1) - \ell_z(\ell_z + 1) = 1.$$

To show that $\sigma_n(z) = \sigma'_n(z)$, note first that neither z nor $z - \mathbf{e}_2$ is in C because $x = 0$. The right neighbor $z + \mathbf{e}_1$ might be in C, but since $\ell_z(\ell_z - 1)$ is even, the last chip sent from $z + \mathbf{e}_1$ by $F^{u'_n}$ does not move to z. The left neighbor $z - \mathbf{e}_1$ is in C only if $\ell_{z-\mathbf{e}_1} = \ell_z - 1 \equiv 3 \bmod 4$, which implies $\ell_z(\ell_z - 1) \equiv 0 \bmod 4$. Hence if $z - \mathbf{e}_1$ is in C, the last chip sent from $z - \mathbf{e}_1$ by $F^{u'_n}$ moves west. It follows that F^{u_n} and $F^{u'_n}$ fire z the same number of times and send the same number of chips to z, hence $\sigma_n(z) = \sigma'_n(z) = 1$.

Case 2: $x > 0$. To argue that $\sigma'_n(z) = 1$, as an initial step we *un*fire every vertex on the positive x-axis $B = \{(m, 0) \in \mathbb{Z}^2 : m > 0\}$ once. Since all initial rotors on B point east, this turns all these rotors north without affecting the number of chips at z (nor at any other vertex of Q).

Now we apply $F^{u'_n}$. By firing the four neighbors of z in parallel, it is easy to see from (2) that they send one chip to z every firing round, since after every round exactly one of their rotors points to z. Hence, firing these neighbors $\ell_z(\ell_z - 1)$ times each sends $\ell_z(\ell_z - 1)$ chips to z. Since $\ell_{z-\mathbf{e}_1} = \ell_{z-\mathbf{e}_2} = \ell_z + 1$, the two neighbors $z - \mathbf{e}_1$ and $z - \mathbf{e}_2$ each fire

$$(\ell_z + 1)(\ell_z + 2) - \ell_z(\ell_z - 1) = 4\ell_z + 2$$

additional times under $F^{u'_n}$. Considering what happens when they are fired in parallel shows that they send one chip to z every two times they fire, meaning that $2\ell_z + 1$ additional chips are sent to z.

Finally, to obtain σ'_n we must fire every vertex in B once more. But since $F^{u'_n}$ fires each vertex in B an even number of times, their rotors are now pointing either north or south, so firing them once more does not affect the number of chips at z. Hence

$$\sigma'_n(z) = \ell_z(\ell_z - 1) + (2\ell_z + 1) - \ell_z(\ell_z + 1) = 1.$$

To finish the proof, we now argue that $\sigma_n(z) = \sigma'_n(z)$. First note that since $\ell_v(\ell_v+1)$ is even for all $v \in D_{n-1}$, it follows from (2) that the last chips sent from $z \pm \mathbf{e}_1$ by $F^{u'_n}$ do not move to z. However, consider the neighbor $z + \mathbf{e}_2$. If $\ell_{z+\mathbf{e}_2} = \ell_z - 1 \equiv 3 \bmod 4$, its final rotor points north after firing $\ell_z(\ell_z - 1)$ times, while if $\ell_z - 1 \equiv 2 \bmod 4$, its final rotor points south. It therefore follows from the definition of C, that F^{u_n} sends one fewer chip from $z + \mathbf{e}_2$ to z than $F^{u'_n}$ in case $\ell_z \equiv 3 \bmod 4$ and $y + 1 \geq 2$. Likewise, F^{u_n} sends one fewer chip from $z - \mathbf{e}_2$ to z than $F^{u'_n}$ in case $\ell_z \equiv 2 \bmod 4$ and $y - 1 \geq 1$. But these are precisely the two cases when $z \in C$, hence F^{u_n} also fires z once fewer than $F^{u'_n}$. Therefore, $\sigma_n(z) = \sigma'_n(z) = 1$. □

We remark that the rotor configuration ρ_n is obtained from ρ'_n by *cycle-popping*: that is, for each directed cycle of rotors in ρ'_n, unfire each vertex in the cycle once. Popping a cycle causes each vertex in the cycle to send one chip to the previous vertex, so there is no net movement of chips. Let ρ''_n be the acyclic rotor configuration obtained from cycle-popping, and let

$$u''_n = u'_n - c_n$$

where $c_n(z)$ is the number of times z is unfired during cycle-popping. Then

$$F^{u''_n}(\rho_0, (2n^2 + 2n + 1)\delta_o) = (\rho''_n, 1_{D_n}).$$

By Lemma 5, we have

$$F^{u_n}(\rho_0, (2n^2 + 2n + 1)\delta_o) = (\rho_n, 1_{D_n}).$$

By the Strong Abelian Property (Theorem 2), it follows that $u''_n = u_n$. In particular, $c_n = 1_C$ and $\rho''_n = \rho_n$.

Proof of Theorem 1. For $m \geq 0$, let r_m be the smallest integer such that $2r_m(r_m + 1) > m$. Consider a modified rotor-router aggregation defined by the growth rule

$$\tilde{A}_{m+1} = \tilde{A}_m \cup \{\tilde{z}_m\}$$

where $\tilde{z}_m$ is the endpoint of a rotor-router walk in $\hat{\mathbb{Z}}^2$ and stopped on exiting $\tilde{A}_m \cap D_{r_m - 1}$. Define $\tilde{u}_n : D_{n-1} \to \mathbb{N}$ by setting

$$\tilde{u}_n(z) = \# \text{ of times } z \text{ fires during the formation of } \tilde{A}_{2n(n+1)}.$$

We will induct on n to show that $u_n = \tilde{u}_n$ for all $n \geq 1$. Since $u_n = \tilde{u}_n$ implies $A_{2n(n+1)} = \tilde{A}_{2n(n+1)} = D_n$ by Lemma 5, this proves the theorem.

The base case of the induction is immediate: $u_1 = \tilde{u}_1 = 4\delta_o$. For $n \geq 2$, in the induced subgraph D_n of $\hat{\mathbb{Z}}^2$ with sink vertices L_n we have

$$F^{u_n}(\rho_0, (2n^2 + 2n + 1)\delta_o) = (\rho_n, 1_{D_n})$$

by Lemma 5. On the other hand,

$$F^{\tilde{u}_n}(\rho_0, (2n^2 + 2n + 1)\delta_o) = (\tilde{\rho}_n, \tilde{\sigma}_n)$$

for some rotor configuration $\tilde{\rho}_n$ on D_{n-1} and chip configuration $\tilde{\sigma}_n$ on D_n. By the inductive hypothesis, $\tilde{A}_{2n(n-1)} = D_{n-1}$, from which it follows that in the formation of $\tilde{A}_{2n(n+1)}$ from $\tilde{A}_{2n(n-1)}$, all rotor-router walks are stopped on exiting D_{n-1}. Together these facts imply that $\tilde{\sigma}_n = 1$ on D_{n-1}. Moreover, since ρ_0 is acyclic, $\tilde{\rho}_n$ is acyclic (each rotor points in the direction a chip last exited). The Strong Abelian Property (Theorem 2) now gives $u_n = \tilde{u}_n$, which completes the inductive step. □

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